Understanding X-nonnegative signomials, by way of X-circuits.

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Presentation to the De Loera Group, UC Davis
August 19, 2020

What are signomials?

Start with "monomial" basis functions, for $\boldsymbol{\alpha} \in \mathbb{R}^n$

$$\mathrm{e}^{\boldsymbol{\alpha}} : \mathbb{R}^n \to \mathbb{R}_{++} \quad \text{takes values} \quad \mathrm{e}^{\boldsymbol{\alpha}}(\boldsymbol{x}) = \exp(\boldsymbol{\alpha}^\intercal \boldsymbol{x}).$$

A signomial is a linear combination

$$f = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} c_{\boldsymbol{\alpha}} \mathrm{e}^{\boldsymbol{\alpha}}.$$

For modeling reasons, signomials are usually written in *geometric form*

$$\boldsymbol{y} \mapsto \sum_{\boldsymbol{\alpha} \in \mathcal{A}} c_{\boldsymbol{\alpha}} \prod_{i=1}^{n} y_i^{\alpha_i} \quad \text{where} \quad y_i = \exp x_i.$$

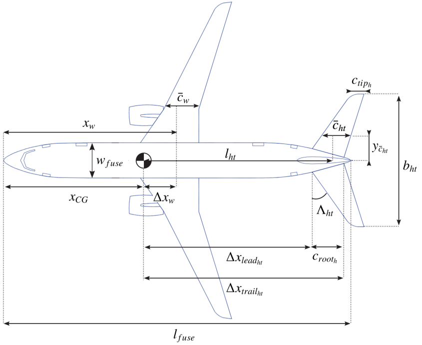
If a signomial has all nonnegative coefficients, call it a *posynomial*.

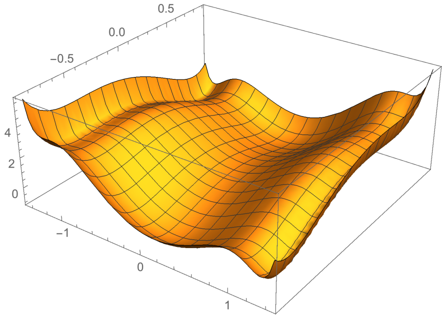
Signomial applications in optimization

A *signomial program* (SP) is an optimization problem stated with signomials, e.g.

$$\inf_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ f(\boldsymbol{x}) \;:\; g_i(\boldsymbol{x}) \leq 1 \text{ for all } i \text{ in } [k] \right\}.$$

Major applications in aircraft design [1, 2, 3, 4, 5] and structrual engineering [6, 7, 8, 9]. Additional applications in EE [10], communications [11], and ML [12].





Signomials are probably useful for you, too!

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Consider the polynomial

$$p(t) = 1 + t - t^3 + t^4.$$

Suitably constructed (geometric-form) signomials can ...

(a) exactly mimic the polynomial on positive reals [13, 14]

$$a(y) = 1 + y - y^3 + y^4.$$

(b) produce a lower-bounding function [15, 16, 17, 18]

$$b(y) = 1 - y - y^3 + y^4.$$

(c) perfectly capture zeroth-order behavior

$$c(y) = 1 + \left(y - \frac{1}{y}\right) - \left(y - \frac{1}{y}\right)^3 + \left(y - \frac{1}{y}\right)^4.$$

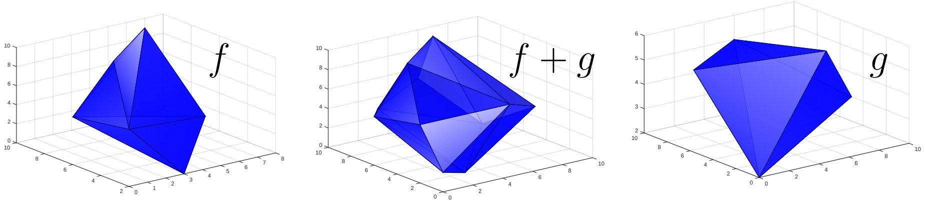
Newton polytopes

Descartes' Rule of Signs: Given real $\alpha_1 < \alpha_2 < \cdots < \alpha_m$, the number of positive roots of

$$y \mapsto c_1 y^{\alpha_1} + c_2 y^{\alpha_2} + \cdots + c_m y^{\alpha_m}$$

is bounded above by the number of sign alternations in $c_1, c_2, \ldots, c_m$.

In higher dimensions, consider "Newton polytopes": convex hulls of exponent vectors.



Infer properties of polynomials over $\mathbb{R}^n_{++}$, signomials over $\mathbb{R}^n$ (remember $y_i = \exp x_i$).

What are $\mathbb{R}^n$-circuits?

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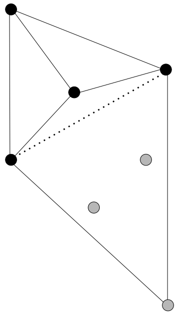
A circuit is a minimal affinely dependent $A \subset \mathbb{R}^n$.

A circuit is simplicial if $\operatorname{conv} A$ has $|A| - 1$ extreme points.



For a finite ground set $\mathcal{A} \subset \mathbb{R}^n$, define

$$\mathbb{R}^{\mathcal{A}} := \{\text{real } |\mathcal{A}|\text{-tuples indexed by } \boldsymbol{\alpha} \in \mathcal{A}\},$$

so we can also understand $\mathcal{A}$ as an operator

$$\mathcal{A} : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^n, \quad \mathcal{A}\boldsymbol{\nu} = \textstyle\sum_{\boldsymbol{\alpha} \in \mathcal{A}} \boldsymbol{\alpha} \nu_{\boldsymbol{\alpha}}.$$

Circuits $A \subset \mathcal{A}$ are in 1-to-1 correspondence with lines

$$\{\boldsymbol{\nu} \in \mathbb{R}^{\mathcal{A}} : \mathbf{1}^\intercal \boldsymbol{\nu} = 0, \text{ and } \operatorname{supp} \boldsymbol{\nu} \text{ minimal among } \boldsymbol{\nu} \in \ker \mathcal{A}\}.$$

For simplicial circuits, $\{\boldsymbol{\alpha} : \nu_{\boldsymbol{\alpha}} > 0\} = \operatorname{ext} \operatorname{conv} A$.

Definitions from convex analysis

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A set convex set K is called a cone if

$$\boldsymbol{x} \in K \quad \Rightarrow \quad \lambda \boldsymbol{x} \in K \qquad \text{for all } \lambda > 0;$$

the dual cone to K is

$$K^* = \{\boldsymbol{y} \,:\, \boldsymbol{y}^\intercal \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \text{ in } K\}.$$

A convex set X induces a support function

$$\sigma_X(\boldsymbol{\lambda}) = \sup\{\boldsymbol{\lambda}^\intercal \boldsymbol{x} \,:\, \boldsymbol{x} \text{ in } X\}.$$

This presentation

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Content from arXiv:2006:06811, joint work with Thorsten Theobald and Helen Naumann.

- Goal of that paper: better understand constrained signomial nonnegativity.
- Along the way: generalize "circuit" to the constrained setting.

Outline of remaining presentation

- Signomial nonnegativity on $X \subset \mathbb{R}^n$ via Sums-of-AM/GM-Exponentials.
- For $X \subset \mathbb{R}^n$ convex and $\mathcal{A} \subset \mathbb{R}^n$ finite, define simplicial X-circuits.
- Explain some properties of these simplicial X-circuits.
- Explain how simplicial X-circuits uniquely construct X-SAGE cones.

Sums of AM/GM-Exponentials

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Definition. An AM/GM-Exponential or "AGE function" is an $\mathbb{R}^n$-nonnegative signomial, which has at most one negative coefficient [19].

Example. Let $f = \mathrm{e}^{\boldsymbol{\alpha}_1} + \mathrm{e}^{\boldsymbol{\alpha}_2} + \mathrm{e}^{\boldsymbol{\alpha}_3} - 3\mathrm{e}^{\boldsymbol{\alpha}_4}$, for

$$\boldsymbol{\alpha}_1 = (2,4,0), \quad \boldsymbol{\alpha}_2 = (4,2,0), \quad \boldsymbol{\alpha}_3 = (0,0,6), \quad \boldsymbol{\alpha}_4 = (2,2,2).$$

Use $\mathrm{e}^{\boldsymbol{\alpha}}(\boldsymbol{x}) = \exp(\boldsymbol{\alpha}^\intercal \boldsymbol{x}) = \mathrm{e}^{\boldsymbol{x}}(\boldsymbol{\alpha})$ and convexity to bound the negative term

$$\boldsymbol{\alpha}_4 = \frac{1}{3}\boldsymbol{\alpha}_1 + \frac{1}{3}\boldsymbol{\alpha}_2 + \frac{1}{3}\boldsymbol{\alpha}_3 \quad \Rightarrow \quad 3\mathrm{e}^{\boldsymbol{\alpha}_4} \leq \mathrm{e}^{\boldsymbol{\alpha}_1} + \mathrm{e}^{\boldsymbol{\alpha}_2} + \mathrm{e}^{\boldsymbol{\alpha}_3}.$$

Historical note on nomenclature: convexity of $\exp$ is equivalent to the AM/GM inequality

$$\boldsymbol{u} \in \mathbb{R}^m_{++}, \ \boldsymbol{\lambda} \in \mathbb{R}^m_{+}, \ \boldsymbol{\lambda}^\intercal \mathbf{1} = 1 \quad \Rightarrow \quad \prod_{i=1}^m u_i^{\lambda_i} \geq \sum_{i=1}^m u_i \lambda_i.$$

SAGE for constrained problems

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An X-**AGE function** is an X-nonnegative signomial, with at most one negative coefficient.

Introduce symbols for AGE cones and SAGE cones

$$C_X(\mathcal{A}) = \sum_{\boldsymbol{\beta}\in\mathcal{A}} \overbrace{\left\{ f \,:\, \boldsymbol{c}_{\setminus\boldsymbol{\beta}} \geq \mathbf{0},\ f = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} c_{\boldsymbol{\alpha}} \mathrm{e}^{\boldsymbol{\alpha}},\ f \geq 0 \text{ on } X \right\}}^{C_X(\mathcal{A},\boldsymbol{\beta})}$$

Theorem ([20])

Fix $\boldsymbol{\beta} \in \mathcal{A}$, and let $f = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} c_{\boldsymbol{\alpha}} \mathrm{e}^{\boldsymbol{\alpha}}$. If X is convex, then $f \in C_X(\mathcal{A},\boldsymbol{\beta})$ if and only if

$$\exists\, \boldsymbol{\nu} \in \mathbb{R}^{\mathcal{A}} \quad \textit{where} \quad \sigma_X(-\mathcal{A}\boldsymbol{\nu}) + D\left(\boldsymbol{\nu}_{\setminus\boldsymbol{\beta}}, e\boldsymbol{c}_{\setminus\boldsymbol{\beta}}\right) \leq c_{\boldsymbol{\beta}} \quad \textit{and} \quad \mathbf{1}^\intercal \boldsymbol{\nu} = 0.$$

Special aspects of $X = \mathbb{R}^n$

Immediate: $\boldsymbol{\nu}$ must satisfy $\mathcal{A}\boldsymbol{\nu} = \mathbf{0}$.

Nontrivial: If f is extremal, then $\operatorname{supp} \boldsymbol{c} \subset \mathcal{A}$ is a simplicial $\mathbb{R}^n$-circuit.

Definition of X-circuits

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For $\boldsymbol{\beta} \in \mathcal{A}$, abbreviate $M_{\boldsymbol{\beta}} = \{\boldsymbol{\nu} \in \mathbb{R}^{\mathcal{A}} \,:\, \mathbf{1}^{\intercal}\boldsymbol{\nu} = 0,\ \boldsymbol{\nu}_{\setminus\boldsymbol{\beta}} \geq \mathbf{0},\ \nu_{\boldsymbol{\beta}} = -1\}$.

Definition. A vector $\boldsymbol{\lambda}^{\star} \in M_{\boldsymbol{\beta}}$ is a normalized X-circuit if

1. $\sigma_X(-\mathcal{A}\boldsymbol{\lambda}^{\star}) < +\infty$, and
2. strict local-sublinearity holds:

   If $L \subset M_{\boldsymbol{\beta}}$ is a segment properly containing $\boldsymbol{\lambda}^{\star}$, then $\sigma_X(-\mathcal{A}\boldsymbol{\lambda})$ is nonlinear on L.

Defined for *any* closed convex set X.

Easy to characterize when X is a cone.

Can't recover $\boldsymbol{\lambda}^{\star}$ given only information on $\operatorname{supp} \boldsymbol{\lambda}^{\star} = \{\boldsymbol{\alpha} : \lambda^{\star}_{\boldsymbol{\alpha}} \neq 0\}$.

Let $\Lambda_X(\mathcal{A}, \boldsymbol{\beta})$ denote circuits in $M_{\boldsymbol{\beta}}$, and aggregate

$$\Lambda_X(\mathcal{A}) = \bigcup_{\boldsymbol{\beta} \in \mathcal{A}} \Lambda_X(\mathcal{A}, \boldsymbol{\beta}).$$

Circuits induced by polyhedral X

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For a face F of a polyhedron P, have the outer normal cone

$$\mathsf{N}_P(F) = \{\boldsymbol{w} \,:\, \boldsymbol{z}^\intercal \boldsymbol{w} = \sigma_P(\boldsymbol{w}) \,\forall\, \boldsymbol{z} \in F\}.$$

The outer normal fan aggregates the normal cones

$$\mathcal{O}(P) = \{\mathsf{N}_P(F) \,:\, F \text{ is a face of } P\}.$$

Normal fans provide a useful decomposition

$$\biguplus_{K \in \mathcal{O}(P)} \operatorname{relint} K = \{\boldsymbol{z} \,:\, \sigma_P(\boldsymbol{z}) < +\infty\}.$$

Theorem ([21])

If X is polyhedral, then every circuit $\boldsymbol{\lambda} \in \Lambda_X(\mathcal{A}, \boldsymbol{\beta})$ generates a ray in the normal fan

$$\mathcal{O}\left(-\mathcal{A}^\intercal X - N_{\boldsymbol{\beta}}^*\right)$$

where $N_{\boldsymbol{\beta}} = \{\boldsymbol{\nu} \in \mathbb{R}^{\mathcal{A}} \,:\, \mathbf{1}^\intercal \boldsymbol{\nu} = 0,\ \boldsymbol{\nu}_{\setminus \boldsymbol{\beta}} \geq \mathbf{0}\}$.

X-circuits for AGE functions

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In order for $f = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} c_{\boldsymbol{\alpha}} \mathrm{e}^{\boldsymbol{\alpha}}$ to be X-AGE with $c_{\boldsymbol{\beta}} < 0$ we need

$$\exists\, \boldsymbol{\nu} \in \mathbb{R}^{\mathcal{A}} \quad \text{where} \quad \sigma_X(-\mathcal{A}\boldsymbol{\nu}) + D(\boldsymbol{\nu}_{\setminus\boldsymbol{\beta}}, e\boldsymbol{c}_{\setminus\boldsymbol{\beta}}) \leq c_{\boldsymbol{\beta}} \quad \text{and} \quad \mathbf{1}^\intercal \boldsymbol{\nu} = 0.$$

Fix $\boldsymbol{\lambda} \in M_{\boldsymbol{\beta}}$, restrict $\boldsymbol{\nu} = s\boldsymbol{\lambda}$ with scale $s \geq 0$, and optimize over s. We get

$$C_X(\mathcal{A}, \boldsymbol{\lambda}) := \left\{ \boldsymbol{c} \in \mathbb{R}^{\mathcal{A}} \,:\, \prod_{\boldsymbol{\alpha} \neq \boldsymbol{\beta}} \left[\frac{c_{\boldsymbol{\alpha}}}{\lambda_{\boldsymbol{\alpha}}} \right]^{\lambda_{\boldsymbol{\alpha}}} \geq -c_{\boldsymbol{\beta}} \exp\left(\sigma_X(-\mathcal{A}\boldsymbol{\lambda})\right) \ \text{ and } \ \boldsymbol{c}_{\setminus\boldsymbol{\beta}} \geq 0 \right\}.$$

Theorem ([21])

If X is a closed convex set, X-AGE cones admit the representation

$$C_X(\mathcal{A}, \boldsymbol{\beta}) = \operatorname{conv}\left(\cup \left\{ C_X(\mathcal{A}, \boldsymbol{\lambda}) \,:\, \boldsymbol{\lambda} \in \Lambda_X(\mathcal{A}, \boldsymbol{\beta}) \right\}\right).$$

Reduced X-circuits and SAGE

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For $\boldsymbol{\lambda} \in \Lambda_X(\mathcal{A})$, denote $\phi_{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}, \sigma_X(-\mathcal{A}\boldsymbol{\lambda})) \in \mathbb{R}^{\mathcal{A}} \times \mathbb{R}$.

We call $\boldsymbol{\lambda}^\star$ a *reduced circuit* if $\phi_{\boldsymbol{\lambda}^\star}$ is an edge generator of

$$\operatorname{cone}\left(\{\phi_{\boldsymbol{\lambda}} \,:\, \boldsymbol{\lambda} \in \Lambda_X(\mathcal{A})\} \cup \{(\mathbf{0}, 1)\}\right).$$

Theorem

Let $R \subset \mathbb{R}^{\mathcal{A}}$ denote the set of reduced circuits for $(\mathcal{A}, X)$. If X is polyhedral, then

$$C_X(\mathcal{A}) = \sum_{\boldsymbol{\lambda} \in R} C_X(\mathcal{A}, \boldsymbol{\lambda})$$

and moreover there is no proper subset $\Lambda \subsetneq R$ for which $C_X(\mathcal{A}) = \sum_{\boldsymbol{\lambda} \in \Lambda} C_X(\mathcal{A}, \boldsymbol{\lambda})$.

Conclusion

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Example lines of future work:

- If X is polyhedral: formal connection to matroid theory?
- Analyze "non-simplicial" X-circuits.
- Define "circuits" for product space of symmetric matrices. E.g.,

$$X \subset \mathbb{S}^n, \qquad \mathcal{A} = \{\boldsymbol{A}_i \in \mathbb{S}^n\}_{i=1}^m, \quad \text{and} \quad \underbrace{(\boldsymbol{V}_i)_{i=1}^m \mapsto \sigma_X\left(-\sum_{i=1}^m \boldsymbol{A}_i \circ \boldsymbol{V}_i\right)}_{\text{require local strict-sublinearity}}.$$

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